On the explicit selection matrix of relating the tensor products and Hadamard products of matrices

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Abstract. In his article [4], Visick gave an explicit expression of the relation between the tensor products and the Hadamard products of two $n \times n$ matrices. We will attempt to generalize Visick's identity to cover the products of finitely many of $m \times n$ matrices.

Key words: Hadamard product, positive unital linear map, selection matrix, tensor product.

1. Introduction

The work in this paper is based on a link between two important matrix products. If $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is an $s \times t$ matrix, then their *tensor (*or *Kronecker) product* is the $ms \times nt$ matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ldots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

If $A = (a_{ij})$ and $B = (b_{ij})$ are both $m \times n$ matrices, then their *Hadamard product* is an $m \times n$ matrix of entry-wise products

$$A \circ B = (a_{ij}b_{ij}).$$

By means of the induction, the tensor product and the Hadamard product of any finitely many matrices can be defined.

Let $\mathbb{C}^n$ be the n dimensional complex vector space and $\mathbb{C}_{n\times n}$ be the vector space of all $n \times n$ complex matrices. Recall that if $A = (a_{ij})$ is an $n \times n$ complex matrix, its *adjoint* A^* has $(i,j)^{th}$ entry $\overline{a_{ji}}$. Any $A \in \mathbb{C}_{n\times n}$ satisfying $A = A^*$ is called *self-adjoint* or *Hermitian.* An Hermitian matrix A is called

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- *positive semidefinite* (written $A \geq 0$) if $(Ax, x) \geq 0$ for all non-zero $x \in \mathbb{C}^n$, and
- *positive definite* (written $A > 0$) if $(Ax, x) > 0$ for all non-zero $x \in \mathbb{C}^n$.

We can partially order the Hermitian $n \times n$ matrices by defining $A_1 \geq A_2$, which means $A_1 - A_2 \geq 0$. This is sometimes called the *Loewner ordering.*

Let H_m be the set of all $m \times m$ Hermitian matrices, and let H_m^+ be the set of all positive definite $m \times m$ matrices. A linear map from H_n into H_m is said to be *positive* if it transforms H_n^+ into H_m^+. A positive linear map Φ from H_n into H_m is said to be *unital* if $\Phi(I_n) = I_m$, where I_k is the $k \times k$ identity matrix. We know [1, Lemma 4, p.224] that there is a unital positive linear map Φ_k from H_{n^k} into H_n such that, for all $n \times n$ matrices A_i $(1 \leq i \leq k)$,

$$\Phi_k \left(\prod_{i=1}^{k} \otimes A_i \right) = \prod_{i=1}^{k} \circ A_i.$$

The interesting question is that if there exists a form of expression which can present the linear map Φ_k.

In February 1997, Visick submitted an article [4], in which he gave an explicit expression of the selection matrix of the relation between the tensor products and the Hadamard products of two $n \times n$ matrices. In order to look into Visick's identity, it is necessary to define, for each $1 \leq i,\ j \leq n$, $E_{ij}^{(n)}$ be the $n \times n$ matrix, which has a single 1 in the $(i,j)^{th}$ position and zeros elsewhere. We then use these matrices to define an $n^2 \times n$ matrix P_n such that

$$P_n^t = \left[E_{11}^{(n)} \ E_{22}^{(n)} \ \ldots \ E_{nn}^{(n)} \right],$$

where P_n^t is the transpose of P_n. Visick's results is

Theorem 1 ([4], Theorem 1) *Let A and B be $m \times n$ matrices. Then*

$$A \circ B = P_m^t (A \otimes B) P_n.$$

In 1999, Mond and Pečarić generalized Visick identity to any finite many $n \times n$ matrices case and gave the following statement [3, Lemma 2.2, p. 57]:

Theorem 2 *Let A_i, $i = 1, ..., k$, be $n \times n$ matrices. There exists an $n^k \times n$ selection matrix P, such that $P^t P = I$ and*

$$P^t \left(\prod_{i=1}^{k} \otimes A_i \right) P = \prod_{i=1}^{k} \circ A_i,$$

where P^t is the transpose of P.

In the result above, the authors didn't give out the explicit expresion of the selection matrix. In this article, our contribution is exhibiting the explicit expresion of the selection matrix for any finite many $m \times n$ matrices about the relation between the tensor products and Hadamard products, and we offer a natural proof, which is different from [3]. As an application, we exhibit some new Hadamard inequalities, which are not follow simply from the work of Visick.

2. Main Result

First we list some useful properties of the tensor product. These properties are mostly known, and can be found in [2].

Lemma 1 ([2], p.15, 1) *Let A_i, B_i be matrices such that A_iB_i $(1 \leq i \leq k)$ are well defined. Then*

$$\left(\prod_{i=1}^{k} \otimes A_i\right)\left(\prod_{i=1}^{k} \otimes B_i\right) = \prod_{i=1}^{k} \otimes (A_iB_i).$$

Lemma 2 ([2], p.15, 3) *Let A_i be $n_i \times n_i$ matrix $(1 \leq i \leq k)$. Then*

$$\left(\prod_{i=1}^{k} \otimes A_i\right)^* = \prod_{i=1}^{k} \otimes A_i^*.$$

Let us introduce the terminology $O^{(n)}$ for the $n \times n$ matrix with all entries equal to 0, and an $n^k \times n$ matrix P_{kn} such that

$$P_{kn}^t = \left[E_{11}^{(n)} O^{(n)} \ldots O^{(n)} E_{22}^{(n)} O^{(n)} \ldots O^{(n)} \ldots O^{(n)} \ldots O^{(n)} E_{nn}^{(n)}\right],$$

where there are $\sum\limits_{l=1}^{k-2} n^l$ zero matrices $O^{(n)}$ between each pair of $E_{ii}^{(n)}$ and $E_{i+1,i+1}^{(n)}$ $(1 \leq i \leq n-1)$.

The main result in this paper is the generalization of Visick's identity:

Theorem 3 *Suppose $k \geq 2$. Let A_i $(1 \leq i \leq k)$ be $m \times n$ matrices. Then*

$$\prod_{i=1}^{k} \circ A_i = P_{km}^t \left(\prod_{i=1}^{k} \otimes A_i\right) P_{kn}.$$

For proving theorem 3, we need the following lemma.

Lemma 3 *Let $A_i = (a_{st}^{(i)})$ $(1 \le i \le k)$ be $n \times m$ matrices. Then $a_{s_1t_1}^{(1)} a_{s_2t_2}^{(2)} \cdots a_{s_kt_k}^{(k)}$ lies in the position of*

$$\left(\sum_{l=1}^{k-1} n^{k-l}(s_l - 1) + s_k, \sum_{l=1}^{k-1} m^{k-l}(t_l - 1) + t_k \right), \qquad (*)$$

in the tensor product of $\prod_{i=1}^{k} \otimes A_i$. Conversely, that entry lies in the position $(*)$ of the matrix $\prod_{i=1}^{k} \otimes A_i$ must be $a_{s_1t_1}^{(1)} a_{s_2t_2}^{(2)} \cdots a_{s_kt_k}^{(k)}$.

Proof. Using induction on k. When $k = 2$

$$A_1 \otimes A_2 = \begin{bmatrix} a_{11}^{(1)} A_2 & \cdots & a_{1m}^{(1)} A_2 \\ \vdots & \cdots & \vdots \\ a_{n1}^{(1)} A_2 & \cdots & a_{nm}^{(1)} A_2 \end{bmatrix}$$

It is easy to check $a_{s_1t_1}^{(1)} a_{s_2t_2}^{(2)}$ lies in the position $(n(s_1 - 1) + s_2, m(t_1 - 1) + t_2)$ of the matrix $A_1 \otimes A_2$. Suppose lemma is true for k, that is $a_{s_1t_1}^{(1)} a_{s_2t_2}^{(2)} \cdots a_{s_kt_k}^{(k)}$ lies in the position

$$\left(\sum_{l=1}^{k-1} n^{k-l}(s_l - 1) + s_k, \sum_{l=1}^{k-1} m^{k-l}(t_l - 1) + t_k \right).$$

By the definition of the tensor product of matrices, we know

$$\prod_{i=1}^{k+1} \otimes A_i = \left(a_{s_1t_1}^{(1)} \cdots a_{s_kt_k}^{(k)} A_{k+1} \right),$$

hence $a_{s_1t_1}^{(1)} \cdots a_{s_kt_k}^{(k)} a_{s_{k+1}t_{k+1}}^{(k+1)}$ lies in the position of

$$\left(n \sum_{l=1}^{k} n^{k-l}(s_l - 1) + s_{k+1}, m \sum_{l=1}^{k} m^{k-l}(t_l - 1) + t_{k+1} \right)$$

$$= \left(\sum_{l=1}^{(k+1)-1} n^{(k+1)-l}(s_l - 1) + s_{k+1}, \sum_{l=1}^{(k+1)-1} m^{(k+1)-l}(t_l - 1) + t_{k+1} \right)$$

in the matrix $\prod_{i=1}^{k+1} \otimes A_i$. By the induction, the lemma is true for all positive integer k. The converse is obvious.

Proof of theorem 3 Since P_{kn}^t is an $n \times n^k$ matrix, $\prod_{i=1}^{k} \otimes A_i$ is an $n^k \times m^k$ matrix and P_{km} is an $m^k \times m$ matrix, $P_{kn}^t (\prod_{i=1}^{k} \otimes A_i) P_{km}$ is an $n \times m$ matrix. For proving the equality in theorem 3, it is enough to check the entry in the position (i, j) of the matrix $P_{kn}^t (\prod_{i=1}^{k} \otimes A_i) P_{km}$ is the element $a_{ij}^{(1)} \cdots a_{ij}^{(k)}$.

In the matrix P_{kn}^t the i^{th} row vector is

$$(0, \cdots, 0, 1, 0, \cdots, 0),$$

in which the number of 1 lies the position of P_{kn}^t is

$$\left(i, n(i-1) + n(i-1)\sum_{l=1}^{k-2} n^l + i\right)$$

$$= \left(i, \sum_{l=1}^{k-1} n^l(i-1) + i\right) = \left(i, \sum_{l=1}^{k-1} n^{k-l}(i-1) + i\right).$$

The entries of the i^{th} row in the matrix $P_{kn}^t(\prod_{i=1}^k \otimes A_i)$ are the entries of the $(\sum\limits_{l=1}^{k-1} n^{k-l}(i-1) + i)^{th}$ row of the matrix $\prod_{i=1}^k \otimes A_i$. The entry in the position (i,j) of the matrix $P_{kn}^t(\prod_{i=1}^k \otimes A_i)P_{km}$ is the sum of the $(\sum\limits_{l=1}^{k-1} n^{k-l}(i-1) + i)^{th}$ row of the matrix $\prod_{i=1}^k \otimes A_i$ multiples the j^{th} column $(0, \cdots, 0, 1, 0, \cdots, 0)^T$ of the matrix P_{km}, in which the number of 1 lies the position is

$$\left(\sum_{l=1}^{k-1} m^l(j-1) + j, j\right) = \left(\sum_{l=1}^{k-1} m^{k-l}(j-1) + j, j\right).$$

Hence the entry in the position of the matrix $P_{kn}^t(\prod_{i=1}^k \otimes A_i)P_{km}$ is the entry in the position

$$\left(\sum_{l=1}^{k-1} n^{k-l}(i-1) + i, \sum_{l=1}^{k-1} m^{k-l}(j-1) + j\right)$$

of the matrix $\prod_{i=1}^k \otimes A_i$. By lemma 3, the entry is exactly $a_{ij}^{(1)} \cdots a_{ij}^{(k)}$. $\square$

3. Applications

To proceed further, we need properties of the P_{kn}'s which are analogous to properties of the P_n's proved in [4, Corollary 3]. We omit the straightforward proof.

Theorem 4 *(i) For any k, $P_{kn}^t P_{kn} = I_n$; and $P_{kn}P_{kn}^t$ is a diagonal $n^k \times n^k$ matrix of zeros and ones satisfying $0 \leq P_{kn}P_{kn}^t \leq I_{n^k}$.*

(ii) For any k, there exists an $n^k \times (n^k - n)$ matrix Q_{kn} of zeros and ones such that the block matrix $[P_{kn}Q_{kn}]$ is an $n^k \times n^k$ permutation matrix and

$$P_{kn}^t Q_{kn} = 0, \quad Q_{kn}^t Q_{kn} = I_{n^k - n}, \quad \text{and} \quad P_{kn}P_{kn}^t + Q_{kn}Q_{kn}^t = I_{n^k}.$$

(iii) For any $m^k \times n^k$ matrix M,

$$0 \leq \left(P_{km}^t M P_{kn}\right)\left(P_{km}^t M P_{kn}\right)^* \leq P_{km}^t M M^* P_{km}.$$

Corresponding Theorem 4 in [4], we have

Theorem 5 *Let A_i $(1 \le i \le k)$ be $m \times n$ matrices. Then*

$$\prod_{i=1}^{k} \circ (A_i A_i^*) = \left(\prod_{i=1}^{k} \circ A_i\right)\left(\prod_{i=1}^{k} \circ A_i\right)^* + \left(P_{km}^t \left(\prod_{i=1}^{k} \otimes A_i\right) Q_{kn}\right)\left(P_{km}^t \left(\prod_{i=1}^{k} \otimes A_i\right) Q_{kn}\right)^* .$$

Proof. From Lemma 1 and Lemma 2,

$$\prod_{i=1}^{k} \otimes \, (A_i A_i^*) = \left(\prod_{i=1}^{k} \otimes A_i\right)\left(\prod_{i=1}^{k} \otimes A_i\right)^* .$$

But by (ii) of Theorem 4,

$$\begin{aligned}
\prod_{i=1}^{k} \otimes (A_i A_i^*) &= \left(\prod_{i=1}^{k} \otimes A_i\right)(P_{kn}P_{kn}^t + Q_{kn}Q_{kn}^t)\left(\prod_{i=1}^{k} \otimes A_i\right)^* \\
&= \left(\prod_{i=1}^{k} \otimes A_i\right)(P_{kn}P_{kn}^t)\left(\prod_{i=1}^{k} \otimes A_i\right)^* + \left(\prod_{i=1}^{k} \otimes A_i\right) Q_{kn}Q_{kn}^t \left(\prod_{i=1}^{k} \otimes A_i\right)^* .
\end{aligned}$$

Successive applications of Theorem 3 lead to

$$\begin{aligned}
\prod_{i=1}^{k} \circ (A_i A_i^*) &= P_{km}^t \left(\prod_{i=1}^{k} \otimes (A_i A_i^*)\right) P_{km} \\
&= \left(P_{km}^t \left(\prod_{i=1}^{k} \otimes A_i\right) P_{kn}\right)\left(P_{km}^t \left(\prod_{i=1}^{k} \otimes A_i\right) P_{kn}\right)^* \\
&\qquad + \left(P_{km}^t \left(\prod_{i=1}^{k} \otimes A_i\right) Q_{kn}\right)\left(Q_{kn}^t \left(\prod_{i=1}^{k} \otimes A_i\right)^* P_{km}\right) \\
&= \left(\prod_{i=1}^{k} \circ A_i\right)\left(\prod_{i=1}^{k} \circ A_i\right)^* + \left(P_{km}^t \left(\prod_{i=1}^{k} \otimes A_i\right) Q_{kn}\right)\left(P_{km}^t \left(\prod_{i=1}^{k} \otimes A_i\right) Q_{kn}\right)^* .
\end{aligned}$$

□

Visick [4, Theorem 11] proved that if A_1 and A_2 are $m \times n$ matrices and $s \in [-1, 1]$, then

$$A_1 A_1^* \circ A_2 A_2^* + s A_1 A_2^* \circ A_2 A_1^* \ge (1 + s)(A_1 \circ A_2)(A_1 \circ A_2)^* .$$

It is possible now to develop it from Theorem 5.

Theorem 6 *For any $m \times n$ matrices A_i $(1 \le i \le k)$ and any real scalars $\alpha_1, \ldots, \alpha_k$, which are not all zeros, then*

$$(\alpha_1^2 + \cdots + \alpha_k^2) \prod_{i=1}^{k} \circ (A_i A_i^*) + \sum_{r=1}^{k-1} w_r \prod_{l=1}^{k} \circ (A_l A_{(l+r)'}^*)$$

$$\ge (\alpha_1 + \cdots + \alpha_k)^2 \left(\prod_{i=1}^{k} \circ A_i \right) \left(\prod_{i=1}^{k} \circ A_i \right)^*,$$

where $w_r = \sum_{l=1}^{k} \alpha_l \alpha_{(l+r)'}$ and $l + r \equiv (l+r)' (\mathrm{mod} k)$ with $0 \le (l+r)' < k$.

Proof. For real scalars $\alpha_1, \ldots, \alpha_k$, which are not all zeros, let

$$M = \alpha_1 (A_1 \otimes \cdots \otimes A_k) + \alpha_2 (A_2 \otimes \cdots \otimes A_k \otimes A_1) + \cdots + \alpha_k (A_k \otimes A_1 \otimes \cdots \otimes A_{k-1}).$$

By Theorem 3 and the symmetry of the Hadamard product,

$$P_{km}^t M P_{kn} = (\alpha_1 + \cdots + \alpha_k) \prod_{i=1}^{k} \circ A_i,$$

and hence, using (iii) of Theorem 4,

$$\begin{aligned} P_{km}^t M M^* P_{km} &\ge \left(P_{km}^t M P_{kn} \right) \left(P_{km}^t M P_{kn} \right)^* \\ &= (\alpha_1 + \cdots + \alpha_k)^2 \left(\prod_{i=1}^{k} \circ A_i \right) \left(\prod_{i=1}^{k} \circ A_i \right)^*. \end{aligned}$$

On the other hand, taking indices mod$k$, Lemma 1 and Lemma 2 give

$$\begin{aligned} M M^* &= \left(\sum_{i=1}^{k} \alpha_i (A_i \otimes A_{i+1} \otimes \cdots \otimes A_{i-1}) \right) \left(\sum_{j=1}^{k} \alpha_j (A_j \otimes A_{j+1} \otimes \cdots \otimes A_{j-1}) \right)^* \\ &= \left(\sum_{i=1}^{k} \alpha_i (A_i \otimes A_{i+1} \otimes \cdots \otimes A_{i-1}) \right) \left(\sum_{j=1}^{k} \alpha_j (A_j^* \otimes A_{j+1}^* \otimes \cdots \otimes A_{j-1}^*) \right) \\ &= \alpha_1^2 (A_1 A_1^* \otimes \cdots \otimes A_k A_k^*) + \cdots + \alpha_k^2 (A_k A_k^* \otimes A_1 A_1^* \otimes \cdots \otimes A_{k-1} A_{k-1}^*) \\ &\quad + \sum_{i \ne j} \alpha_i \alpha_j (A_i A_j^* \otimes A_{i+1} A_{j+1}^* \otimes \cdots \otimes A_{i-1} A_{j-1}^*). \end{aligned}$$

Now another application of Theorem 3 and the commutativity of the Hadamard product yield

$$P_{km}^t MM^* P_{km} = (\alpha_1^2 + \cdots + \alpha_k^2) \prod_{i=1}^k \circ (A_i A_i^*) + \sum_{r=1}^{k-1} w_r \prod_{l=1}^k \circ (A_l A_{(l+r)'}^*),$$

where $w_r = \sum_{l=1}^k \alpha_l \alpha_{(l+r)'}$ and $l + r \equiv (l+r)' (\mathrm{mod} k)$ with $0 \leq (l+r)' < k$. Consequently,

$$(\alpha_1^2 + \cdots + \alpha_k^2) \prod_{i=1}^k \circ (A_i A_i^*) + \sum_{r=1}^{k-1} w_r \prod_{l=1}^k \circ (A_l A_{(l+r)'}^*)$$
$$\geq (\alpha_1 + \cdots + \alpha_k)^2 \left(\prod_{i=1}^k \circ A_i \right) \left(\prod_{i=1}^k \circ A_i \right)^* .$$

□

We start by examining some special cases briefly in order to see, firstly, if we set $\alpha_1 = 1$ and $\alpha_2 = \cdots = \alpha_k = 0$, Theorem 6 really is an extension of

$$\prod_{i=1}^k \circ (A_i A_i^*) \geq \left(\prod_{i=1}^k \circ A_i \right) \left(\prod_{i=1}^k \circ A_i \right)^* .$$

Next, we recover the result [4, Theorem 11] which we mentioned before the statement of Theorem 6. Let $k = 2$, and for every $s \in [-1, 1]$, let $\alpha_1 = 1$ and $\alpha_2 = \alpha$ be a real solution of the equation

$$s\alpha^2 - 2\alpha + s = 0,$$

so that

$$s = \frac{2\alpha}{1 + \alpha^2}.$$

Then Theorem 6 asserts that

$$(1 + \alpha^2)(A_1 A_1^*) \circ (A_2 A_2^*) + 2\alpha \; (A_1 A_2^*) \circ (A_2 A_1^*) \geq (1 + \alpha)^2 (A_1 \circ A_2)(A_1 \circ A_2)^*.$$

A simplification gives

$$(A_1 A_1^*) \circ (A_2 A_2^*) + s(A_1 A_2^*) \circ (A_2 A_1^*) \geq (1 + s)(A_1 \circ A_2)(A_1 \circ A_2)^*.$$

Finally, we present an attractive inequality for three matrices. This does not follow simply from the work of Visick. If $k = 3$ and $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = -\frac{1}{2}$, Theorem 6 asserts that

$$(A_1 A_1^*) \circ (A_2 A_2^*) \circ (A_3 A_3^*) \geq \frac{1}{2} [(A_1 A_2^*) \circ (A_2 A_3^*) \circ (A_3 A_1^*) + (A_2 A_1^*) \circ (A_3 A_2^*) \circ (A_1 A_3^*)].$$

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